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Convergence acceleration of some logarithmic sequences

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Abstract: Let (S_n) be some real sequence defined as

$$\begin{aligned} &S_{n+1}=f(S_n) \quad \text{for } n\in\mathbb{N}, \text{ where } f(x)=x+\sum_{i\geqslant 1}\alpha_{p+i}(x-S)^{p+i}, \\ &\quad \text{with } p\in\mathbb{N}, \quad p\neq 0 \quad \text{and} \quad \alpha_{p+1}<0, \qquad S_0 \text{ given.} \end{aligned} \tag{1}$$

For S_0 well chosen, it converges to S and $\lim_{n\to+\infty}(S_{n+1}-S)/(S_n-S)=1$ (logarithmic convergence). By asymptotic analysis, we show that different algorithms, modified iterated versions of Aitken's Δ^2 process, iterated θ_2-algorithm, modified ϵ-algorithms and θ-algorithm, accelerate the convergence of this type of sequences and we estimate the errors on the transformed sequences. For sequences of type (1) with $p=1$, 2, we give more precise results and an efficient algorithm combining the modified Δ^2 and θ_2-algorithm. Finally, we apply these algorithms to some series and integrals.

Keywords: Convergence acceleration, logarithmic sequences, integrals, series.

1. Introduction

Let f be a function whose asymptotic expansion (a.e.) around 0 is:

$$f(x)=x+\sum_{i\geqslant 1}\alpha_{p+i}x^{p+i}, \quad p\geqslant 1 \quad \text{and} \quad \alpha_{p+1}<0.$$

We denote by A_p the set of these functions and we define:

$$\mathrm{LOGF}_p=\left\{(x_n):\ x_{n+1}=f(x_n),\ f\in A_p \text{ and } \lim_{n\to+\infty} x_n=0\right\}.$$

Theorem 1 (Sedogbo [10]). *If* $(x_n)\in\mathrm{LOGF}_p$, *then we have*

$$x_n=\mathrm{O}(n^{-1/p}).$$

2. Asymptotic expansions of the sequences of LOGF_p; definition of subsets of A_p and LOGF_p for $p = 1,2$

According to de Bruijn [3, Chapter 8], there exists, for each $f \in A_p$, a unique function ψ,

$$\psi(x) = \sum_{i=1}^{p} d_i x^{-i} + c_0 \log x + \sum_{i \geqslant 1} c_i x^i, \tag{2}$$

verifying the functional equation

$$\psi(f(x)) = 1 + \psi(x). \tag{3}$$

Setting $x_0 = x$ and $x_n = f(x_{n-1}) = f_n(x)$, where $f_1 = f$ and $f_n = f \circ f_{n-1}$ for $n \geqslant 2$, we deduce from (3):

$$\psi(x_n) = \psi(f_n(x)) = n + \psi(x).$$

Then the coefficients of ψ can be computed and we obtain the asymptotic expansion of x_n for (x_n) element of LOGF_p, $p \geqslant 1$.

Let us recall the results given in [9] for $p = 1,2$.

Theorem 2. *For* $f \in A_1$,

$$\psi(x) = d_1 x^{-1} + c_0 \log x + c_1 x + c_2 x^2 + \cdots.$$

Let

$$A_1' = \{ f \in A_1\colon c_0 \neq 0 \} \quad \textit{and} \quad \mathrm{LOGF}_1' = \{ (x_n)\colon f \in A_1' \},$$
$$A_1'' = \{ f \in A_1\colon c_0 = 0 \} \quad \textit{and} \quad \mathrm{LOGF}_1'' = \{ (x_n)\colon f \in A_1'' \}.$$

(i) *For* $(x_n) \in \mathrm{LOGF}_1'$, *the a.e. is*:

$$x_n = a_1 n^{-1} + (a_2 \log n + a_3) n^{-2} + \left(a_4 (\log n)^2 + a_5 \log n + a_6\right) n^{-3} + \cdots.$$

(ii) *For* $(x_n) \in \mathrm{LOGF}_1''$, *the a.e. is*:

$$x_n = a_1 n^{-1} + a_3 n^{-2} + a_6 n^{-3} + a_{10} n^{-4} + \cdots.$$

Theorem 3. *For* $f \in A_2$,

$$\psi(x) = d_2 x^{-2} + d_1 x^{-1} + c_0 \log x + c_1 x + c_2 x^2 + \cdots.$$

Let

$$A_2' = \{ f \in A_2\colon \alpha_4 \neq 0 \textit{ and } c_0 \neq 0 \} \quad \textit{and} \quad \mathrm{LOGF}_2' = \{ (x_n)\colon f \in A_2' \},$$
$$A_2'' = \{ f \in A_2\colon f \textit{ odd},\ c_0 \neq 0 \} \quad \textit{and} \quad \mathrm{LOGF}_2'' = \{ (x_n)\colon f \in A_2'' \},$$
$$A_2''' = \{ f \in A_2\colon f \textit{ odd},\ c_0 = 0 \} \quad \textit{and} \quad \mathrm{LOGF}_2''' = \{ (x_n)\colon f \in A_2''' \}.$$

(i) *For* $(x_n) \in \mathrm{LOGF}_2'$, *the a.e. is*:

$$x_n = a_1 n^{-1/2} + a_2 n^{-1} + (a_3 \log n + a_4) n^{-3/2} + (a_5 \log n + a_6) n^{-2} + \cdots.$$

(ii) *For* $(x_n) \in \mathrm{LOGF}_2''$, *the a.e. is*:

$$x_n = a_1 n^{-1/2} + (a_3 \log n + a_4) n^{-3/2} + \left(a_7 (\log n)^2 + a_8 \log n + a_9\right) n^{-5/2} + \cdots.$$

(iii) *For* $(x_n) \in \mathrm{LOGF}_2'''$, *the a.e. is*:

$$x_n = a_1 n^{-1/2} + a_4 n^{-3/2} + a_9 n^{-5/2} + \cdots.$$

3. General algorithms for $(x_n) \in \mathrm{LOGF}_p$

We propose a specific study of some algorithms which accelerate the convergence of sequences in LOGF_p and also estimate the corresponding errors on the transformed sequences.

These algorithms are: some suitable modifications of the ϵ-algorithm and the iterated Aitken's Δ^2 process, the iterated θ_2-algorithm and the θ-algorithm.

The results given by the four algorithms are very similar.

Modified ϵ-algorithm

Modifying the ϵ-algorithm [2] leads us to the next algorithm.

Algorithm 1. $\epsilon_{-1}^{(n)} = 0$ and for $(k, n) \in \mathbb{N}^2$, $\epsilon_0^{(n)} = x_n$, $\epsilon_{k+1}^{(n)} = \epsilon_{k-1}^{(n+1)} + A_k[\Delta\epsilon_k^{(n)}]^{-1}$, where (A_k) depends only on the subset containing (x_n), we have the next theorem.

Theorem 4 (Sedogbo [10]). *The convergence of each sequence $(x_n) \in \mathrm{LOGF}_p$ can be accelerated by the modified ϵ-algorithm with*

$$A_{-1} = 0, \qquad A_{2k} = \frac{1}{k+1}, \quad A_{2k+1} = k + p + 1, \qquad \textit{for } k \in \mathbb{N}.$$

More precisely, for $k \in \mathbb{N}$, $(\epsilon_{2k+2}^{(n)})$ converges faster than $(\epsilon_{2k}^{(n)})$ and $\epsilon_{2k}^{(n)} = \mathrm{O}(x_n^{k+1}) = \mathrm{O}(n^{-(k+1)/p})$.

Modified iterated Δ^2 algorithm

Given any sequence (u_n), we define

$$R[u_n] = \frac{\Delta u_n \cdot \Delta u_{n+1}}{\Delta^2 u_n}.$$

Thus, the classical Δ^2 algorithm [2] applied to (u_n) gives

$$v_n = u_{n+1} - R[u_n].$$

Algorithm 2. For each sequence (x_n), we define

$$x_n^{(0)} = x_n \quad \text{for } n \in \mathbb{N} \text{ and for } (k, n) \in \mathbb{N}^2,$$
$$x_n^{(k+1)} = x_{n+1}^{(k)} - B_{k+1} R[x_n^{(k)}];$$

(B_k) depends only on the subset containing (x_n).

Theorem 5 (Sedogbo [10]). *Applying Algorithm 2 to $(x_n) \in \mathrm{LOGF}_p$ with $B_k = (k+p)/k$, $(x_n^{(k)})$ converges faster than $(x_n^{(k-1)})$ for $k \geqslant 1$. Moreover, $x_n^{(k)} = \mathrm{O}(x_n^{k+1}) = \mathrm{O}(n^{-(k+1)/p})$, for $k \in \mathbb{N}$.*

Iterated θ_2-algorithm

The θ_2-algorithm [2] applied to (u_n) can be described as follows:

$$\hat{\theta}(u_n) = u_{n+1} + (u_{n+1} - v_n)\frac{\Delta u_{n+1}}{\Delta u_{n+1} - \Delta v_n},$$

where $v_n = u_{n+1} - R[u_n]$.

Let

$$\hat{\theta}_k(u_n) = \hat{\theta}\big(\hat{\theta}_{k-1}(u_n)\big) \quad \text{for } k \geqslant 1, \qquad \big(\hat{\theta}_0(u_n) = u_n\big).$$

Theorem 6 (Sedogbo [10]). *For $(x_n) \in \mathrm{LOGF}_p$, and for $k \in \mathbb{N}$, $(\hat{\theta}_{k+1}(x_n))$ converges faster than $(\hat{\theta}_k(x_n))$; moreover, $\hat{\theta}_k(x_n) = \mathrm{O}(x_n^{k+1}) = \mathrm{O}(n^{-(k+1)/p})$.*

θ-algorithm

The θ-algorithm [2] applied to (u_n) can be described as follows: $\theta_{-1}^{(n)} = 0$, $\theta_0^{(n)} = u_n$ for $n \in \mathbb{N}$ and

$$\theta_{2k+2}^{(n)} = \theta_{2k}^{(n+1)} + \frac{\Delta\theta_{2k}^{(n+1)}}{1 - \dfrac{\Delta\theta_{2k+1}^{(n)}}{\Delta\theta_{2k+1}^{(n+1)}}},$$

$\theta_{2k+1}^{(n)} = \theta_{2k-1}^{(n+1)} + [\theta_{2k}^{(n+1)} - \theta_{2k}^{(n)}]^{-1}$, for $(k, n) \in \mathbb{N}^2$.

Theorem 7 (Sedogbo [10]). *For $(x_n) \in \mathrm{LOGF}_p$, and for $k \in \mathbb{N}$, $(\theta_{2k+2}^{(n)})$ converges faster than $(\theta_{2k}^{(n)})$, moreover, $\theta_{2k}^{(n)} = \mathrm{O}(x_n^{k+1}) = \mathrm{O}(n^{-(k+1)/p})$.*

4. The case of LOGF_p, $p = 1,2$

For $p = 1,2$ we have more precise results, therefore more efficient algorithms.

4.1. Modified ϵ-algorithm, modified iterated Δ^2 algorithm, iterated θ_2-algorithm

Theorem 8 (Sedogbo [10]). (i) *For $(x_n) \in \mathrm{LOGF}_p'$, Algorithm 1 with $A_{-1} = 0$, $A_{2k} = 1/(k+1)$, $A_{2k+1} = k + p + 1$, $k \in \mathbb{N}$, gives:*

$$\epsilon_{2k}^{(n)} = \mathrm{O}\big(x_n^{k+1}\big) = \mathrm{O}\big(n^{-(k+1)/p}\big).$$

(ii) *For $(x_n) \in \mathrm{LOGF}_p''$, Algorithm 1 with $A_{-1} = 0$, $A_{2k} = 1/(2k+1)$, $A_{2k+1} = 2k + p + 1$, $k \in \mathbb{N}$, gives:*

$$\epsilon_{2k}^{(n)} = \mathrm{O}\big(x_n^{2k+1}\big) = \mathrm{O}\big(n^{-(2k+1)/p}\big).$$

(iii) *For $(x_n) \in \mathrm{LOGF}_2'''$, Algorithm 1 with $A_{-1} = 0$, $A_{2k} = 1/(4k+1)$, $A_{2k+1} = 4k + 3$, $k \in \mathbb{N}$, gives:*

$$\epsilon_{2k}^{(n)} = \mathrm{O}\big(x_n^{4k+1}\big) = \mathrm{O}\big(n^{-(4k+1)/2}\big).$$

Theorem 9 (Sablonniere [9]). *Algorithm 2 gives the same results as above with:*

(i) $B_k = (k+p)/k$, *for* $(x_n) \in \mathrm{LOGF}_p'$,
(ii) $B_k = (2k+p-1)/(2k-1)$, *for* $(x_n) \in \mathrm{LOGF}_p''$,
(iii) $B_k = (4k-1)/(4k-3)$, *for* $(x_n) \in \mathrm{LOGF}_2'''$.

Theorem 10 (Sablonniere [9]). *For each subset of* LOGF_p, $p=1,2$, *the iterated θ_2-algorithm gives the same asymptotic results as those of Theorem* 8.

4.2. Combinations of the modified Δ^2-algorithm and θ_2-algorithm

Combining the modified Δ^2 algorithm with the θ_2-algorithm, the results are more efficient than those of the previous section.

Algorithm 3. For each sequence (x_n) we define

$$x_n^{(0)}=x_n \quad \text{for } n\in\mathbb{N} \text{ and for } k\geqslant 1,\ n\in\mathbb{N},$$
$$y_n^{(k)}=x_{n+1}^{(k-1)}-c_kR\left[x_n^{(k-1)}\right],$$
$$Z_n^{(k)}=\hat{\theta}\left(x_n^{(k-1)}\right),$$
$$x_n^{(k)}=\gamma_k y_n^{(k)}+\lambda_k Z_n^{(k)}.$$

The following theorem completes the results of [9, Theorem 5].

Theorem 11 (Sedogbo [10]). (i) *For* $(x_n)\in\mathrm{LOGF}_p'$, *Algorithm* 3, *with* $c_k=(2k+p-1)/(2k-1)$, $\gamma_k=1/(2k)$, $\lambda_k=(2k-1)/(2k)$, $k\geqslant 1$, *gives*:

$$x_n^{(k)}=\mathrm{O}\left(x_n^{2k+1}\right)=\mathrm{O}\left(n^{-(2k+1)/p}\right).$$

(ii) *For* $(x_n)\in\mathrm{LOGF}_1''$, *Algorithm* 3, *with* $c_k=(3k-1)/(3k-2)$, $\gamma_k=2/(3k)$, $\lambda_k=(3k-2)/(3k)$, $k\geqslant 1$, *gives*:

$$x_n^{(k)}=\mathrm{O}\left(x_n^{3k+1}\right)=\mathrm{O}\left(n^{-(3k+1)}\right).$$

(iii) *For* $(x_n)\in\mathrm{LOGF}_2''$, *Algorithm* 3, *with* $c_k=(4k-1)/(4k-3)$, $\gamma_k=2/(4k-1)$, $\lambda_k=(4k-3)/(4k-1)$, $k\geqslant 1$, *gives*:

$$x_n^{(k)}=\mathrm{O}\left(x_n^{4k+1}\right)=\mathrm{O}\left(n^{-(2k+1/2)}\right).$$

(iv) *For* $(x_n)\in\mathrm{LOGF}_2'''$, *Algorithm* 3, *with* $c_k=(5k-1)/(5k-3)$, $\gamma_k=4/(5k+1)$, $\lambda_k=(5k-3)/(5k+1)$, $k\geqslant 1$, *gives*:

$$x_n^{(k)}=\mathrm{O}\left(x_n^{5k+2}\right)=\mathrm{O}\left(n^{-(5k+2)/2}\right).$$

5. Applications

When the asymptotic expansion of a sequence is known its convergence can be accelerated by a general algorithm of Section 3 or by an algorithm of Section 4.

In [3–7,11,12], one finds series and integrals whose asymptotic expansions are the same as or similar to those of sequences in LOGF_p, thus our algorithms can be applied to them. But, for sequences which have important applications we can construct more specific algorithms, for instance, as shown in the next theorems.

Theorem 12 (Sedogbo [10]). *The convergence of a sequence* (S_n) *verifying*

$$S_n - S = \frac{\beta_1}{n^{1/p}} + \frac{\beta_2}{n^{2/p}} + \frac{\beta_3}{n^{3/p}} + \cdots, \quad \text{for } p \in \mathbb{R},\ p > 0,\ p \neq 1,$$

can be accelerated by the modified ϵ*-algorithm (Algorithm 1) with* $A_{-1} = 0$, $A_{2k} = 1/(k+1)$, $A_{2k+1} = k + p + 1$, *for* $k \in \mathbb{N}$.

More precisely, $(\epsilon_{2k+2}^{(n)})$ *converges faster than* $(\epsilon_{2k}^{(n)})$ *and*

$$\epsilon_{2k}^{(n)} - S = O(n^{-(k+1)/p}).$$

Via the Euler–Maclaurin formula, this theorem allows the convergence acceleration of Riemann series.

Theorem 13 (Sedogbo [10]). *Let us consider the integral* $I = \int_a^b f(x)\,\mathrm{d}x$ *where* f *is a function with derivatives up to some order on* $[a;\ b]$. *We define* (S_n) *as*

$$S_1 = \tfrac{1}{2}h_1(f(a) + f(b)),$$

$$S_n = \tfrac{1}{2}h_n[f(a) + 2f(x_1) \cdots + 2f(x_{n-1}) + f(b)] \quad \text{for } n \geq 2,$$

with $h_i = (b-a)/i$ *and* $x_i = a + ih_i$ *for* $i \geq 1$. *Algorithm 3 applied to* (S_n) *with* $c_k = 3k/(3k-1)$, $\gamma_k = 2/(3k+1)$, $\lambda_k = (3k-1)/(3k+1)$, $k \geq 1$, *accelerates the convergence of* (S_n) *to* I.

More precisely, for $k \in \mathbb{N}$, $(x_n^{(k+1)})$ *converges faster than* $(x_n^{(k)})$ *and* $x_n^{(k)} - I = O(n^{-(3k+2)})$.

The main interest of this theorem is that it estimates the error on the transformed sequence.

6. Numerical results

In this section we give numerical results in order to compare the different algorithms. The sequence that we consider is (X_n): $X_{n+1} = F(X_n)$ with $F(X) = X + \cos X - 1$ and $X_0 = 0.3$.

We have chosen a sequence converging to zero, to avoid rounding errors.

The results are shown in Tables 1 and 2.

Table 1
$(x_n) \in \mathrm{LOGF}_1'$

x_0	$0.3000000000000000D+00$
x_1	$0.2553364891256060D+00$
x_2	$0.2229148521969890D+00$
x_3	$0.1981720495453590D+00$
x_4	$0.1786001474797286D+00$
x_5	$0.1626934912220265D+00$
x_6	$0.1494880718125438D+00$
x_7	$0.1383355217736914D+00$
x_8	$0.1287824126647018D+00$
x_9	$0.1205014122262228D+00$
x_{10}	$0.1132498981151394D+00$
x_{11}	$0.1068439794099804D+00$
x_{12}	$0.1011415892404690D+00$
x_{13}	$0.9603113743139027D-01$
x_{14}	$0.9142369020174071D-01$

Table 2

	Modified ϵ-algorithm	Modified iterated Δ^2-algorithm	Iterated θ_2-algorithm	Combination modified Δ^2 and θ_2
First step	$0.1876092235521526 \cdot 10^{-01}$	$0.1876092235521526 \cdot 10^{-01}$	$-0.2219567031643613 \cdot 10^{-01}$	$-0.1717373980610432 \cdot 10^{-02}$
	$0.1397632449058170 \cdot 10^{-01}$	$0.1397632449058170 \cdot 10^{-01}$	$-0.1603599725302749 \cdot 10^{-01}$	$-0.1029836381222893 \cdot 10^{-02}$
	$0.1086861863505311 \cdot 10^{-01}$	$0.1086861863505311 \cdot 10^{-01}$	$-0.1221142583408038 \cdot 10^{-01}$	$-0.6714035995136342 \cdot 10^{-03}$
	$0.8721490068013468 \cdot 10^{-02}$	$0.8721490068013468 \cdot 10^{-02}$	$-0.9650207706217891 \cdot 10^{-02}$	$-0.4643588191022116 \cdot 10^{-03}$
	$0.7169130280556853 \cdot 10^{-02}$	$0.7169130280556853 \cdot 10^{-02}$	$-0.7840458310750889 \cdot 10^{-02}$	$-0.3356640150970181 \cdot 10^{-03}$
	$0.6006851959700221 \cdot 10^{-02}$	$0.6006851959700221 \cdot 10^{-02}$	$-0.6509095772025902 \cdot 10^{-02}$	$-0.2511219061628403 \cdot 10^{-03}$
	$0.5112061781628142 \cdot 10^{-02}$	$0.5112061781628142 \cdot 10^{-02}$	$-0.5498311771793318 \cdot 10^{-02}$	$-0.1931249950825877 \cdot 10^{-03}$
	$0.4407345417304034 \cdot 10^{-02}$	$0.4407345417304034 \cdot 10^{-02}$	$-0.4711192040969978 \cdot 10^{-02}$	$-0.1519233118329720 \cdot 10^{-03}$
	$0.3841695752986852 \cdot 10^{-02}$	$0.3841695752986852 \cdot 10^{-02}$	$-0.4085289933256996 \cdot 10^{-02}$	$-0.1217970901350721 \cdot 10^{-03}$
	$0.3380309043444033 \cdot 10^{-02}$	$0.3380309043444033 \cdot 10^{-02}$	$-0.3578767606132120 \cdot 10^{-02}$	$-0.9922928134404348 \cdot 10^{-04}$
	$0.2998730780909460 \cdot 10^{-02}$	$0.2998730780909460 \cdot 10^{-02}$	$-0.3162672749661845 \cdot 10^{-02}$	$-0.8197098437619249 \cdot 10^{-04}$
	$0.2679343487201669 \cdot 10^{-02}$	$0.2679343487201669 \cdot 10^{-02}$	$-0.2816415070142678 \cdot 10^{-02}$	$-0.6853579147050459 \cdot 10^{-04}$
	$0.2409178535037678 \cdot 10^{-02}$	$0.2409178535037678 \cdot 10^{-02}$		
Second step	$0.8153494917958511 \cdot 10^{-03}$	$0.6757159365988751 \cdot 10^{-03}$	$0.4881145791152769 \cdot 10^{-03}$	$-0.1817107475191165 \cdot 10^{-05}$
	$0.5445995113648157 \cdot 10^{-03}$	$0.4488781826347245 \cdot 10^{-03}$	$0.3139207830338858 \cdot 10^{-03}$	$-0.9448160395049599 \cdot 10^{-06}$
	$0.3841689250134300 \cdot 10^{-03}$	$0.3154098556952440 \cdot 10^{-03}$	$0.2147391936506407 \cdot 10^{-03}$	$-0.5381251842883406 \cdot 10^{-06}$
	$0.2822684300830688 \cdot 10^{-03}$	$0.2310636908129551 \cdot 10^{-03}$	$0.1537880890303934 \cdot 10^{-03}$	$-0.3279324257874136 \cdot 10^{-06}$
	$0.2140871103426955 \cdot 10^{-03}$	$0.1748445208156351 \cdot 10^{-03}$	$0.1141309664683630 \cdot 10^{-03}$	$-0.2105625582403654 \cdot 10^{-06}$
	$0.1665729412520412 \cdot 10^{-03}$	$0.1357849182618638 \cdot 10^{-03}$	$0.8715259451639739 \cdot 10^{-04}$	$-0.1409539576013813 \cdot 10^{-06}$
	$0.1323570573102981 \cdot 10^{-03}$	$0.1077261529628568 \cdot 10^{-03}$	$0.6812885688033142 \cdot 10^{-04}$	$-0.9762671470394662 \cdot 10^{-07}$
	$0.1070411229060222 \cdot 10^{-03}$	$0.8700784003493591 \cdot 10^{-04}$	$0.5431154684777739 \cdot 10^{-04}$	$-0.6956652813422331 \cdot 10^{-07}$
	$0.8787860690379680 \cdot 10^{-04}$	$0.7135218016169458 \cdot 10^{-04}$	$0.4402335145013490 \cdot 10^{-04}$	$-0.5078019273395959 \cdot 10^{-07}$
	$0.7308894186902055 \cdot 10^{-04}$	$0.5928673683735622 \cdot 10^{-04}$		
	$0.6148063821794949 \cdot 10^{-04}$	$0.4982860980896596 \cdot 10^{-04}$		
Third step	$0.2183508408707539 \cdot 10^{-04}$	$0.1653601278212705 \cdot 10^{-04}$	$-0.2496573487565032 \cdot 10^{-05}$	$-0.2493208490244404 \cdot 10^{-08}$
	$0.1313859645301012 \cdot 10^{-04}$	$0.9844002751073464 \cdot 10^{-05}$	$-0.1296405344017129 \cdot 10^{-05}$	$-0.9083417657464470 \cdot 10^{-09}$
	$0.8450326925189373 \cdot 10^{-05}$	$0.6279806077206122 \cdot 10^{-05}$	$-0.7521015566072393 \cdot 10^{-06}$	$-0.3333975982135785 \cdot 10^{-09}$
	$0.5713425765557122 \cdot 10^{-05}$	$0.4218740709987864 \cdot 10^{-05}$	$-0.4726225570478878 \cdot 10^{-06}$	$-0.1118711539105697 \cdot 10^{-09}$
	$0.4017025785282422 \cdot 10^{-05}$	$0.2950880833895214 \cdot 10^{-05}$	$-0.3156220444325977 \cdot 10^{-06}$	$-0.3336481812619738 \cdot 10^{-10}$
	$0.2914912100115899 \cdot 10^{-05}$	$0.2132246675234340 \cdot 10^{-05}$	$-0.2205358681070660 \cdot 10^{-06}$	$0.3159289991681967 \cdot 10^{-11}$
	$0.2171106958835793 \cdot 10^{-05}$	$0.1582574706357105 \cdot 10^{-05}$		
	$0.1653035645375965 \cdot 10^{-05}$	$0.1201359885619346 \cdot 10^{-05}$		
	$0.1282460111293628 \cdot 10^{-05}$	$0.9296714810341323 \cdot 10^{-06}$		

7. Conclusions

For convenience, we have studied sequences in $LOGF_p$ which converge to zero. But our algorithms and results are also valid for sequences whose limit is not zero. In [10] are given: the proofs of the theorems, a procedure for characterizing a sequence in order to apply the corresponding algorithms and also numerical examples illustrating the algorithms.

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